Volume of a Solid of Revolution

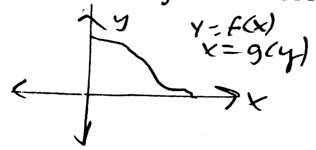
6.1: Disk and Washer Method 6.3: Shell Method

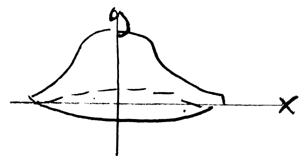
Animations of these examples can be found at
http://www.pccmathuyekawa.com/classes-taught/math_5a/handouts/index.html

The problem: Find the volume of a solid formed by revolving a plane region about an axis.

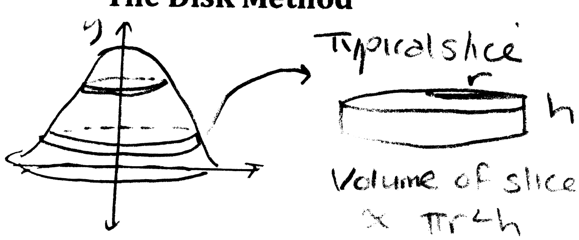
About the y axis:

The idea: given the continuous function and the region shown below. Revolve the region about the y axis to form the solid shown below.





The Disk Method

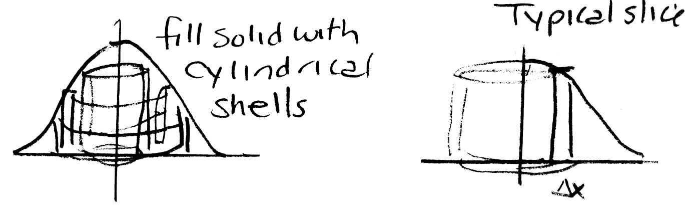


Take slices perpendicular to the y axis. They will be circular disks of volume ΔV ="$\pi r^2 h$". The value of r, and h will vary dependent on the axis of revolution and on the region.

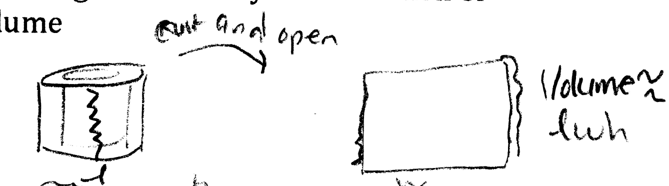
For this case,
r=radius of disk= $g(y)$
h=thickness of disk= Δy

So $\Delta V = \pi[g(y)]^2 \Delta y$
Sum these and take the limit as $n \to \infty$ to get: $V = \int_{ystart}^{ystop} \pi[g(y)]^2 dy$

The Shell Method



Take a rectangle parallel to the y axis. If that rectangle is revolved about the y axis it generates a cylindrical shell of volume



ΔV ="$2\pi r$(*height of rectangle*)(*thickness*)"
The value of r, height, and thickness will vary dependent on the axis of revolution and on the region. For this case:
r= rad. of cylin.=dist from rect to axis= x
height of rectangle= $f(x)$
thickness= Δx

So $\Delta V = 2\pi x f(x) \Delta x$

Sum these up and take the limit as $n \to \infty$ to get $V = \int_{xstart}^{xstop} 2\pi x f(x) dx$

It is better to understand the general idea than to memorize the formulas.

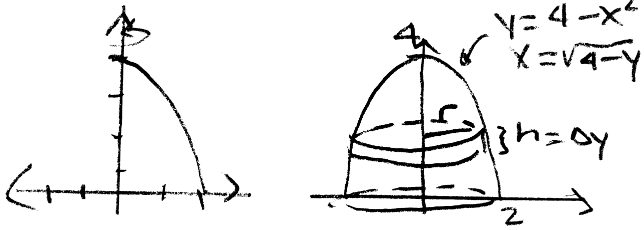
Example: Find the volume when the region bounded by $y = 4 - x^2$ and the y axis, $0 \le x \le 2$ is revolved about the y axis.

The Disk Method

$\Delta V = "\pi r^2 h"$

Draw solid

Draw typical slice, Perpendicular to axis of revolution



r is functional value, x in terms of y.

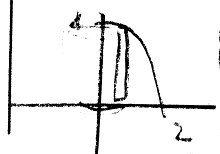
$$V = \int_0^4 \pi (\sqrt{4-y})^2 \, dy$$

The Shell Method

$\Delta V = "2\pi r(\textit{height of rectangle})(\textit{thickness})"$

Draw region only. Drawing whole solid sometimes causes confusion with limits

Draw a rectangle parallel to axis of revolution



$r = x$

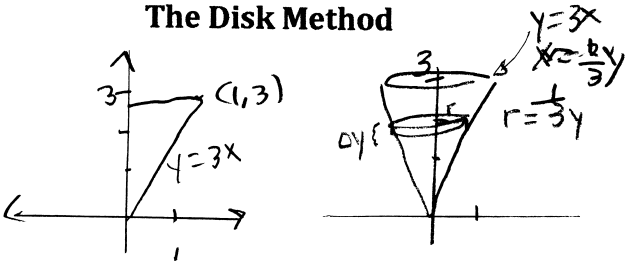
$h = f(x) = 4 - x^2$

thickness = Δx

$$V = \int_0^2 2\pi x(4 - x^2)\, dx$$

Example: Find the volume when the region bounded by $y = 3x$ and $y = 3$ and the y axis, is revolved about the y axis.

The Disk Method

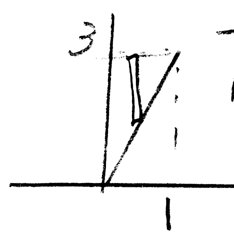


$$V = \int "\pi r^2 h" = \int_0^3 \pi \left(\frac{1}{3}y\right)^2 dy$$

$$= \frac{\pi}{9} \left. \frac{y^3}{3} \right]_0^3$$

$$= \pi$$

The Shell Method



Thickness = Δx

r = dist to y axis = x

h = Top curve − Bottom curve

3 − 3x

$$V = \int "2\pi r h" \text{ thickness}$$

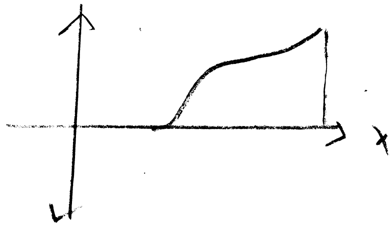
$$= \int_0^1 2\pi x \,[3 - 3x]\, dx$$

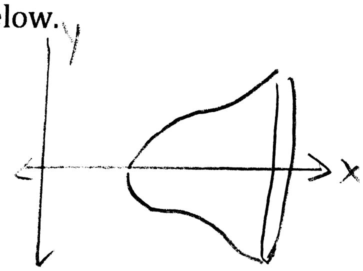
$$= 2\pi \int_0^1 (3x - 3x^2)\, dx$$

$$= 2\pi \left(\frac{3}{2}x^2 - x^3 \right]_0^1 = 2\pi\left(\frac{1}{2}\right) = \pi$$

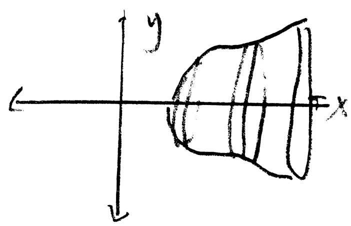
About the x axis:

The idea: given the continuous function and the region shown below. Revolve the region about the x axis to form the solid shown below.





The Disk Method



This time take slices perpendicular to the x axis. They will be circular disks of volume $\Delta V = "\pi r^2 h"$ as before but this time

r=radius of disk= $f(x)$

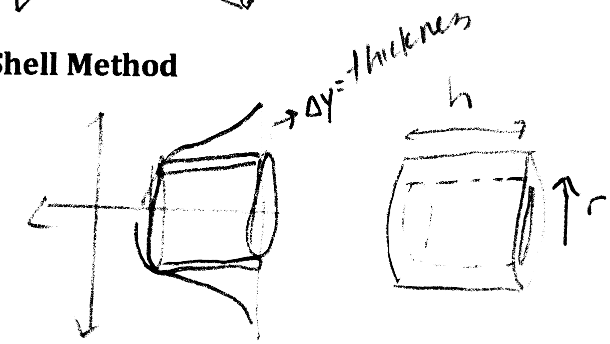
h=thickness of disk= Δx

so $\Delta V = \pi[f(x)]^2 \Delta x$

Sum these and take the limit to get:

$$V = \int_{xstart}^{xstop} \pi[f(x)]^2 dx$$

The Shell Method



Take a rectangle parallel to the x axis. If that rectangle is revolved about the x axis it generates a cylindrical shell of volume $\Delta V = "2\pi r h(thickness)"$ as before but this time

r= rad. of cylin.=dist from rect to axis= y

height of rectangle= $g(y)$

thickness= Δy

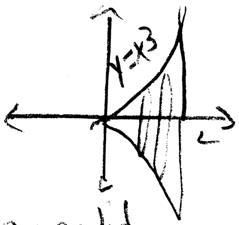
So $\Delta V = 2\pi\, y\, g(y) \Delta y$

Sum these and take the limit to get

$$V = \int_{ystart}^{ystop} 2\pi\, y\, g(y)\, d\cancel{x}y$$

Example: Find the volume when the region bounded by $y = x^3$ and $x = 2$ and the x axis is revolved about the x axis.

The Disk Method



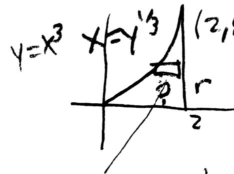
$V = \int \pi r^2 h$

$= \int_0^2 \pi (x^3)^2 \, \Delta x$

$= \pi \left. \frac{x^7}{7} \right]_0^2 = \frac{2^7 \pi}{7}$

Draw solid. Slice perp to x-axis

The Shell Method



$V = \int 2\pi r h$ thickness

$= 2\pi \int^8 y(2 - y^{1/3})\, dy$

ht of rectangle is right curve − left curve

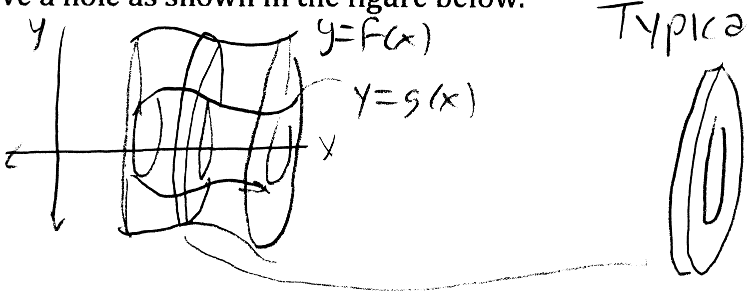
$= 2\pi \int_0^8 (2y - y^{4/3}) dy$

$= 2\pi \left. \left(y^2 - \frac{3}{7} y^{7/3}\right) \right]_0^8$

Washer Method

This is a version of the disk method that needs to be applied when the circular disk slices have a hole as shown in the figure below.

Typical Slice



In this case we take slices perpendicular to the axis of revolution to obtain washer shaped objects, The volume of one slice would be

$\Delta V =$ Volume of outer cylinder − Volume of inner cylinder

$= \pi (\text{outer radius})^2 h - \pi (\text{inner radius})^2 h$

$= \pi [(\text{outer radius})^2 - (\text{inner radius})^2] h$

$\Delta V =$

For revolution about the x axis, outer radius is the top curve, inner radius is the bottom curve. For revolution about the y axis, outer radius is the right curve, inner radius is the left curve. Here outer radius $= f(x)$, inner $= g(x)$

Sum and take the limit, leads to an integral.

$$\pi \int_{x \text{ start}}^{x \text{ stop}} \left([f(x)]^2 - [g(x)]^2 \right) dx$$

But once again..

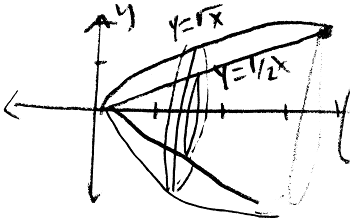
It is better to understand the general idea than to memorize the formulas.

Example: : Find the volume of the solid generated when the region bounded by $y=\sqrt{x}$ and $y=\frac{1}{2}x$ is revolved about the x axis.

$y=\frac{1}{2}x \Rightarrow x=2y$

$y=\sqrt{x} \Rightarrow x=y^2$

The Disk Method – Washers



(4,2) Slice perpendicular to x-axis

outer radius is $y=\sqrt{x}$

Inner radius is $y=\frac{1}{2}x$

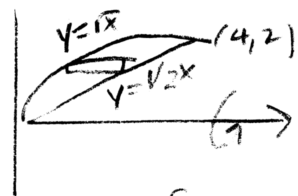
$$V=\int\left[\pi(\text{outer})^2-(\text{inner})^2\right]dx$$

$$=\pi\int_0^4\left[(\sqrt{x})^2-\left(\tfrac{1}{2}x\right)^2\right]dx$$

$$=\pi\int_0^4\left(x-\tfrac{1}{4}x^2\right)dx=\pi\left(\frac{x^2}{2}-\frac{x^3}{12}\right]_0^4$$

$$\pi\left(8-\tfrac{64}{12}\right)=8\pi/3$$

The Shell Method



Draw rectangle parallel to x axis

$$V=\int 2\pi r h\,(\text{thickness})$$

h = Right − left ($x=2y$, $x=y^2$)

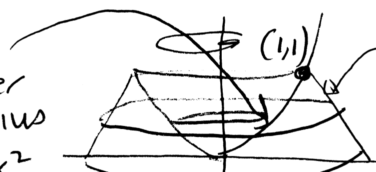
$$=\int_0^2 2\pi y\left[2y-y^2\right]dy$$

$$=2\pi\int_0^2(2y^2-y^3)dy=2\pi\left(\frac{2}{3}y^3-\frac{y^4}{4}\right]_0^2$$

$$=2\pi\left(\frac{16}{3}-4\right)=2\pi\left(\tfrac{4}{3}\right)=\frac{8\pi}{3}$$

Example: Find the volume of the solid generated when the region bordered by $y=x^2$, $y=2-x$ and the x axis is revolved about the y axis.

The Disk Method – Washers



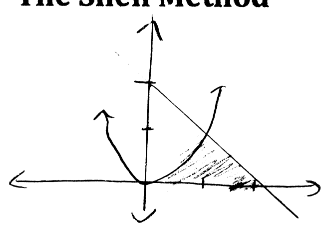
Inner radius $y=x^2 \Rightarrow x=\sqrt{y}$

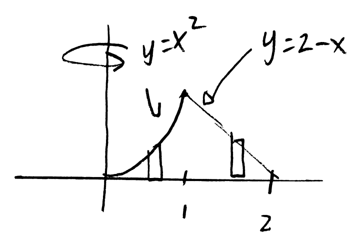
outer radius $y=2-x$, $x=2-y$

$$(\text{outer})^2-(\text{inner})^2$$

$$\pi\int_0^1\left[(2-y)^2-(\sqrt{y})^2\right]dy$$

The Shell Method





Notice---the height of the rectangle is given by the upper curve, but it changes so split.

$2\pi r h$ thickness

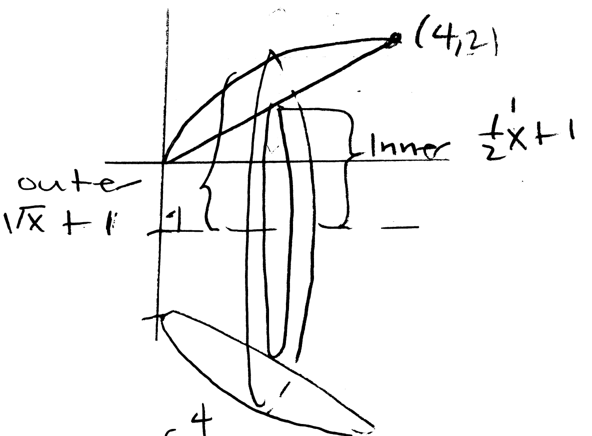
$$\int_0^1 2\pi x(x^2)dx+\int_1^2 2\pi x(2-x)dx$$

About an axis other than the x or y axis

Example: Find the volume of the solid generated when the region bounded by $y=\sqrt{x}$ and $y=\frac{1}{2}x$ is revolved about the line $y=-1$.

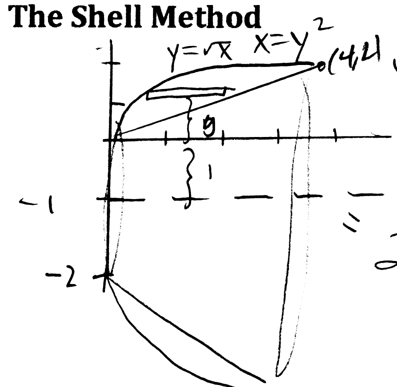
The Disk Method – Washers	The Shell Method

The Disk Method – Washers



$$V=\pi\int_0^4\left[(\sqrt{x}+1)^2-\left(\tfrac{1}{2}x+1\right)^2\right]dx$$

The Shell Method



" $2\pi r h$ "

Rt − Left

y^2-2y

What is affected here is r the distance from the rectangle to the axis of revolution $r=y+1$

So $V=\int_0^2 2\pi(y+1)(y^2-2y)\,dy$

Summary

The Disk/Washer Method

Take slices perpendicular to the axis of revolution. If the slice is a solid disc then the volume can be found by integrating

$$"\pi r^2 h"$$

where h is Δx or Δy and r is the radius of the disk.

If the slice is a washer then integrate
$\pi\left[(outer\ radius)^2-(inner\ radius)^2\right]h$
where h is Δx or Δy.

The Shell Method

Draw a rectangle parallel to the axis of revolution. It will generate an cylindrical shell when revolved. The volume can be found by integrating
$"2\pi r h"$
where r is the distance from the rectangle to the axis of revolution and h is the height of the rectangle.